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The four-element discrete nonlinear Schrödinger equation—non-integrability and Arnold diffusion

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Abstract. We study the integrability of the discrete nonlinear Schrödinger equation with four elements. Using the two-element system as the underlying integrable subsystem we treat the coupling to the additional oscillators perturbationally. A symmetry reduction method is used to map the coupled system of four elements on a three-degree of freedom Hamiltonian system in the reduced phase space. We analyse the geometrical structure of the phase space of the resulting reduced system focusing on the homoclinic manifold. By means of Wiggins' generalized Melnikov method we prove non-integrability through the existence of chaotic dynamics in demonstrating the presence of Arnold diffusion for the coupled dynamics.

1. Introduction

This paper is based on the discrete nonlinear Schrödinger equation (DNLS)

$$\mathrm{i}\frac{\mathrm{d}c_n}{\mathrm{d}t} = -\gamma\,|c_n|^2 c_n - [W_{n,n+1}c_{n+1} + W_{n,n-1}\,c_{n-1}] \tag{1}$$

where the c_n are complex probability amplitudes at site n, γ is the local nonlinearity parameter and $W_{n,n\pm1}$ are the dispersion parameters determining the linear coupling between the sites. This equation arises as a discretization of the continuum nonlinear Schrödinger equation. While the latter is completely integrable [1, 2], numerical studies of the dynamics of its discrete version (1) point to non-integrability (see, e.g. [3, 4]). In fact, in a previous paper the existence of horseshoe chaos in the dynamics of the three-element DNLS has been proven analytically [5] and, to our knowledge, no (rigorous) proof of the existence of chaotic (non-integrable) dynamics for the DNLS with $n > 3$ has been given so far.

Equation (1) is also known as the discrete self-trapping (DST) system and presents a set of coupled nonlinear classical oscillators, introduced by Eilbeck *et al* [6] as a model to describe the nonlinear vibrational dynamics in small polyatomic aggregates. The DST system can also be applied to model the self-trapping phenomena in chemical, condensed matter and optical systems [7–10].

The regular and irregular dynamical behavior of the DNLS system with a few degrees of freedom ($n \leqslant 4$) was investigated in [6, 11–19]. Most of these studies were performed for the DST system with three degrees of freedom, i.e. a trimer, which is the first non-trivial case beyond the dimer case. The latter is integrable and its solutions can be expressed in terms of Jacobian elliptic functions [7, 20].

In this paper we study the integrability of the four-element DNLS, referred to as a DNLS tetramer. The aim of this paper is to prove the existence of chaotic solutions analytically.

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Starting from the integrable dimer and treating the weak coupling to the remaining sites of a tetramer configuration perturbationally, we will prove the existence of homoclinic tangles in the tetramer dynamics by applying Melnikov's method [21,22], further developed by Holmes and Marsden [23] and generalized by Wiggins [24] to cover multidimensional systems as well.

The paper is organized as follows. Section 2 presents the model for the DNLS tetramer and deals with a symmetry reduction to a Hamiltonian system in a lower-dimensional phase space. In order to apply the generalized Melnikov method, in section 3 we investigate the geometrical structure of the phase space with special attention to the homoclinic manifold. Section 4 is devoted to the calculation of the generalized Melnikov function and discussion of the appearence of Arnold diffusion in the coupled oscillator system. Finally, we give a short summary in section 5.

2. The DNLS tetramer and symmetry reduction

We write the DNLS-tetramer Hamiltonian in terms of the complex site-occupation amplitudes (in the following referred to as amplitudes) as

$$H = -\tfrac{1}{2}\gamma \sum_{j=1}^{4} |c_j|^4 - W_{34}(c_4^* c_3 + c_3^* c_4) - \epsilon \left\{ W_{12}(c_2^* c_1 + c_1^* c_2) + W_{13}(c_3^* c_1 + c_1^* c_3) \right.$$
$$\left. + W_{23}(c_3^* c_2 + c_2^* c_3) + W_{14}(c_4^* c_1 + c_1^* c_4) + W_{24}(c_4^* c_2 + c_2^* c_4) \right\} . \qquad (2)$$

Whereas the system is assumed to be symmetric in the real nonlinearity parameters $\gamma_j = \gamma$, it is asymmetric in the real dispersion parameters, $W_{ij} = W_{ji}$ (the bonds in the classical coupled-oscillator picture) by choosing a much stronger coupling $W_{34} \equiv V$ between sites 3 and 4 than between any of the remaining sites. We call the thereby marked (3, 4) pair a *dimer*, the dynamics of which will be taken into account rigorously. The smallness of all couplings other than the intra-dimer coupling V has been made explicit by the dimensionless parameter $0 < \epsilon \ll 1$ in front of the terms which will later be treated perturbationally. This situation can be encountered in studies of dimerized molecular aggregates in which only two of the constituents couple strongly to form the above-mentioned dimer. Different tetramer configurations, as displayed in figure 1, follow from (2) by choosing appropriate sets of dispersion parameters as indicated. To maintain full flexibility the present analysis is based on the most general tetramer configuration (*a*). Since a term $E c_j$ proportional to the on-site energy E can be eliminated from the equations of motion by a simple gauge transformation $c_j \to c_j \exp(-\mathrm{i}Et)$ a corresponding diagonal contribution has been omitted in the Hamiltonian (2).

The system obtained by setting $\epsilon = 0$ is integrable, since it decomposes into the DNLS dimer and the two completely uncoupled single sites 1 and 2 with conserved occupation probabilities. As already mentioned, we are interested in the effect of the small couplings introduced by the perturbing part of the Hamiltonian (2). Instead of explicitly writing the equations of motion for the amplitudes c_j of the system, in which, besides energy conservation, the norm

$$P = \sum_{j=1}^{4} |c_j|^2 \qquad (3)$$

is a further conserved quantity, we first express the Hamiltonian (2) in a different form resulting from:

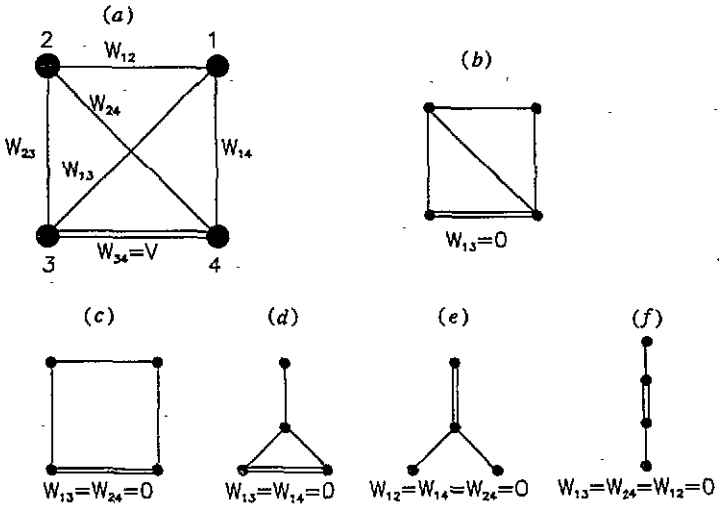

Figure 1. Different tetramer configurations specified by sets of dispersion parameters as indicated.

(i) a transformation of c_1 and c_2 to action-angle variables

$$c_j = \sqrt{J_j}\, e^{-i\theta_j} \qquad (j = 1, 2) \tag{4}$$

and

(ii) a symplectic transformation of the dimer amplitudes c_3 and c_4 to two pairs of canonically-conjugate variables (p, ϕ) and (N, β)

$$c_3 = \sqrt{\frac{N+p}{2}}\, e^{-i(\frac{\phi}{2}+\beta)} \qquad c_4 = \sqrt{\frac{N-p}{2}}\, e^{i(\frac{\phi}{2}-\beta)}. \tag{5}$$

The momentum variable p is equal to the occupation-probability difference at the dimer sites, i.e. $p = |c_3|^2 - |c_4|^2$. The auxiliary action variable N is given by $N = |c_3|^2 + |c_4|^2$ and, obviously, specifies the amount of P contained in the dimer, since $P = N + J_1 + J_2$.

The Hamiltonian (2) expressed in these new variables becomes

$$H = H_d^0(p, \phi; N, \beta) + H_1^0(J_1) + H_2^0(J_2) + \epsilon H^1(p, \phi, N, \beta, J_1, \theta_1, J_2, \theta_2) \tag{6}$$

with

$$\begin{aligned} H^0 &= H_d^0 + \{H_1^0 + H_2^0\} \\ &= -\tfrac{1}{4}\gamma\left[N^2 + p^2\right] - V\sqrt{N^2 - p^2}\cos\phi - \tfrac{1}{2}\gamma\left(J_1^2 + J_2^2\right) \end{aligned} \tag{7}$$

and

$$\begin{aligned} H^1 = -\Bigg\{ & 2W_{12}\sqrt{J_1 J_2}\cos(\theta_1 - \theta_2) + \sum_{j=1,2}\Bigg[W_{3j}\sqrt{2J_j(N+p)}\cos\left(\frac{\phi}{2} + \beta - \theta_j\right) \\ & + W_{4j}\sqrt{2J_j(N-p)}\cos\left(\frac{\phi}{2} - \beta + \theta_j\right)\Bigg]\Bigg\} - \theta_4). \end{aligned} \tag{8}$$

The original complex variables c_j span the eight-dimensional product space $C \times C \times C \times C$, which reduces to a dimension of six as a result of the two conserved quantities H and P. A reduction of the dimensionality of the phase space corresponding to the system of Hamiltonian (6) is possible by applying the symmetry reduction method of Marsden and

Weinstein [25,26]. Apparently, the interaction Hamiltonian depends on the angle variables β and θ_j ($j = 1, 2$) only in their combinations $[\theta_j - \beta]$, which suggests passing to new angle variables $\psi_j = (\theta_j - \beta)$ ($j = 1, 2$) and introducing new couples of canonically conjugate action-angle variables (I_j, ψ_j) and $(\hat{I}_j, \hat{\psi}_j)$ with the generating function

$$F = \sum_{j=1,2} \left[(\theta_j - \beta)I_j + (\theta_j + \beta)\hat{I}_j\right]. \tag{9}$$

The transformed perturber and interaction Hamiltonians (H_1^0, H_2^0) and H^1 are, respectively, given by

$$H_j^0 = -\tfrac{1}{8}\gamma\,(I_j + \hat{I}_j)^2 \qquad (j = 1, 2) \tag{10}$$

and

$$H^1 = -W_{12}\sqrt{(I_1 + \hat{I}_1)(I_2 + \hat{I}_2)}\cos(\psi_1 - \psi_2) - \sum_{i=1,2}\left[W_{3i}\sqrt{(I_i + \hat{I}_i)(N + p)}\,\cos\left(\frac{\phi}{2} - \psi_i\right)\right.$$
$$\left. + W_{4i}\sqrt{(I_i + \hat{I}_i)(N - p)}\,\cos\left(\frac{\phi}{2} + \psi_i\right)\right]. \tag{11}$$

The angle variables $\hat{\psi}_j$ ($j = 1, 2$) are cyclic variables for the Hamiltonian. The corresponding conserved action variables $\hat{I}_j$, and the action N are related by

$$N = \hat{I}_1 + \hat{I}_2 - P\,. \tag{12}$$

In the following N is treated as a system parameter. The reduced three-degrees of freedom Hamiltonian

$$H = H_d^0(p, \phi) + H_1^0(I_1) + H_2^0(I_2) + \epsilon\, H^1(p, \phi, I_1, I_2, \psi_1, \psi_2) \tag{13}$$

written in terms of the final variables is the basis of our explicit treatment. It aims at proving that integrability of the unperturbed system is lost due to the coupling of the dimer to the other tetramer sites. The dynamics of the DNLS dimer exhibits a homoclinic structure for a nonlinearity parameter $\gamma > 2V$ [20] and the following investigation deals with the establishment of chaotic motion in the vicinity of this homoclinic structure as a result of the coupled-tetramer dynamics.

3. The unperturbed system and homoclinic manifolds

In this section we collect the main results for the geometrical structure and the dynamics of the unperturbed system, which is needed to apply the generalized Melnikov method to the perturbed DNLS dimer in section 4.

The integrable equations of motion in the dimer variables (p, ϕ) are given by

$$\dot{p} = -2V\sqrt{N^2 - p^2}\,\sin\phi \tag{14}$$

$$\dot{\phi} = -\gamma\, p + 2V\,\frac{p}{\sqrt{N^2 - p^2}}\,\cos\phi\,. \tag{15}$$

The system (14) and (15) has a hyperbolic fixed point at $p = 0$, $\phi = 0$ for $\alpha = N\gamma/(2V) > 1$. The hyperbolic point is connected to itself by a pair of homoclinic orbits formed by its coinciding stable and unstable manifolds.

The coordinates of the homoclinic orbit, for $2 > \alpha > 1$, are given by

$$p_h^{\pm}(t) = \pm p_h(t) = \frac{4V}{\gamma}\sqrt{\alpha - 1}\,\mathrm{sech}\left[2\sqrt{\alpha - 1}\,Vt\right] \tag{16}$$

$$\phi_h^{\pm}(t) = \phi_h(t) = \arccos\left[\frac{N - \frac{\gamma}{4V}(p_h(t))^2}{\sqrt{N^2 - (p_h(t))^2}}\right]. \tag{17}$$

The six-dimensional phase space of the unperturbed system is given by

$$(p, \phi, I_1, \psi_1, I_2, \psi_2) \in \left(\Re^1 \times T^1 \times \Re^1 \times T^1 \times \Re^1 \times T^1\right). \tag{18}$$

For $\alpha > 1$ the unperturbed system has an invariant four-dimensional normally hyperbolic manifold

$$\mathcal{M}_0 = \{(p, \phi, I_1, \psi_1, I_2, \psi_2) \in \Re^1 \times T^1 \times \Re^1 \times T^1 \times \Re^1 \times T^1 \mid p = 0, \phi = 0, (I_1, I_2) \in \Re^2, (\psi_1, \psi_2) \in [0, 2\pi)\} \tag{19}$$

which has the structure of a two-parameter family of 2-tori. Furthermore, $\mathcal{M}_0$ has five-dimensional stable and unstable manifolds denoted by $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$, respectively. These manifolds coincide along the five-dimensional homoclinic orbits $\Gamma^{\pm}$, which can be parametrized by

$$\Gamma^{\pm} = \left[\left(p_h^{\pm}(-t_0), \phi_h(-t_0), (I_1, I_2), (\psi_1^0, \psi_2^0)\right) \in \Re^1 \times T^1 \times \Re^2 \times T^2 \mid t_0 \in \Re\right]. \tag{20}$$

The completely integrable unperturbed vector field restricted to $\mathcal{M}_0$ is determined by

$$\dot{I}_j = 0 \qquad \dot{\psi}_j = -\tfrac{1}{4}\gamma\,(I_j + \hat{I}_j) \qquad (j = 1, 2) \tag{21}$$

with flow given by

$$I_j(t) = I_j(0) = \text{constant} \qquad \psi_j(t) = -\tfrac{1}{4}\gamma(I_j(0) + \hat{I}_j)\,t + \psi_j^0. \tag{22}$$

The 2-torus on $\mathcal{M}_0$ corresponding to fixed actions $\bar{I} = (\bar{I}_1, \bar{I}_2)$ is denoted by $\mathcal{T}_0(\bar{I})$. This 2-torus has three-dimensional stable and unstable manifolds, $W^s(\mathcal{T}_0(\bar{I}))$ and $W^u(\mathcal{T}_0(\bar{I}))$ which intersect along a three-dimensional homoclinic orbit $\Gamma_{\bar{I}}^{\pm}$, obtained from (20) by fixing the I component.

4. The generalized Melnikov method and Arnold diffusion

For the perturbed system, i.e. $\epsilon \neq 0$, the stable and unstable manifold of the homoclinic manifold may intersect transversely leading to the creation of homoclinic tangles. The analysis of the dynamics in the vicinity of the homoclinic orbit is based on the properties of the generalized multidimensional Melnikov function introduced by Wiggins [24], which provides a signed measure of the distance between the stable and unstable manifolds. In this sense the Melnikov method serves to prove that the integrability will be broken under small perturbations inducing chaotic motions near the homoclinic manifold.

We start with a brief discussion of the geometrical structure of the perturbed phase space. According to Fenichel's invariant manifold theory [28], the homoclinic manifold $\mathcal{M}_0$ persists for sufficiently small $\epsilon \neq 0$ as a locally invariant manifold $\mathcal{M}_\epsilon$ in the perturbed system and has local stable and unstable manifolds $W^{s,u}(\mathcal{M}_\epsilon)$ close to $W^{s,u}(\mathcal{M}_0)$ (for details of the persistance theory for normally hyperbolic manifolds see chapter 4 of Wiggins [24]). Furthermore, under the assumption of non-degeneracy for the frequencies of the decoupled oscillators, i.e. $\det\left[D_I^2 H^0(I)\right] \neq 0$, the KAM theorem [29] allows to conclude that for sufficiently small ϵ most of the non-resonant 2-tori are preserved. For fixed actions $\bar{I}$, a two-dimensional invariant torus $\mathcal{T}_\epsilon(\bar{I})$ is located on $\mathcal{M}_\epsilon$. The existence of the two-dimensional

stable and unstable manifolds, $W^s(\mathcal{T}_\epsilon(\bar{I}))$ and $W^u(\mathcal{T}_\epsilon(\bar{I}))$, of the torus is ensured by Graff's theorem [30].

To determine whether the stable and unstable manifolds $W^s(\mathcal{T}_\epsilon)$ and $W^s(\mathcal{T}_\epsilon)$ of the torus intersect transversely on $\mathcal{M}_\epsilon$, we have to compute the distance d_ϵ between the local stable and unstable manifold at points along the unperturbed homoclinic orbits $\Gamma_{\bar{I}}^{\pm}$. The two-component Melnikov vector $M_{\bar{I}}(\psi_1^0, \psi_2^0) = (M_1(\psi_1^0), M_2(\psi_2^0))$ is proportional to the signed distance in $W^s(\mathcal{T}_\epsilon)(\bar{I})$ and $W^u(\mathcal{T}_\epsilon)(\bar{I})$ to order ϵ which is determined by $(I_\epsilon^u)_j - (I_\epsilon^s)_j$, $(j = 1, 2)$.

According to Wiggins [24], the components of the Melnikov vector are given by the following two Melnikov integrals:

$$M_j^{\pm}(\psi_j^0) = -\int_{-\infty}^{\infty} \mathrm{d}t\, D_{\psi_j} H^1\left[p_h^{\pm}(t), \phi_h(t); \bar{I}, \psi_j(t) + \psi_j^0\right] \qquad (j = 1, 2) \tag{23}$$

which has to be evaluated along the homoclinic orbit of the unperturbed system. We restrict the analysis to the case marked by the '+' sign in (23), since the '−' sign case can be treated in an analogous manner.

Using equation (11) we obtain after some algebra

$$\begin{aligned} M_j^+(\psi_j^0) = -\epsilon\sqrt{\bar{I}_j + \hat{I}_j} \int_{-\infty}^{\infty} \mathrm{d}t\, \cos(\omega_j t)\big\{ & W_{4j}\left[S_-(t)\cos(\psi_j^0) + S_+(t)\sin(\psi_j^0)\right] T_-(t) \\ & - W_{3j}\left[S_-(t)\cos(\psi_j^0) - S_+(t)\sin(\psi_j^0)\right] T_+(t)\big\} \\ & + \epsilon\,(-1)^j\, W_{12}\sqrt{(\bar{I}_1 + \hat{I}_1)(\bar{I}_2 + \hat{I}_2)} \int_{-\infty}^{\infty} \mathrm{d}t\, \sin(\omega_{12} t + (\psi_1^0 - \psi_2^0)) \end{aligned} \tag{24}$$

with

$$S_\pm(t) = \sqrt{\sqrt{N^2 - [p_h(t)]^2} \pm (N - \gamma/(4V)\,[p_h(t)]^2)} \tag{25}$$

$$T_\pm(t) = \left[\frac{N \pm p_h(t)}{N \mp p_h(t)}\right]^{1/4} \tag{26}$$

$$\omega_j = \tfrac{1}{4}\gamma(\bar{I}_j + \hat{I}_j) \qquad \omega_{12} = -\tfrac{1}{4}\gamma\,[(\bar{I}_1 - \bar{I}_2) + (\hat{I}_1 - \hat{I}_2)]\,. \tag{27}$$

The Melnikov integral in (24) is only conditionally convergent. Following an approach given in chapter 4 of [24], we therefore consider the improper integral as the following limit of a sequence in time $\{\tau_n^{s,u}\} = \{\pm 2\pi n/\omega_j\}$, $n = 1, 2, \ldots$:

$$\begin{aligned} M_j^+(\psi_j^0) = -\lim_{n\to\infty} \epsilon\sqrt{\bar{I}_j + \hat{I}_j} \int_{\frac{-2\pi n}{\omega_j}}^{\frac{2\pi n}{\omega_j}} \mathrm{d}t\, \cos(\omega_j t)\big\{ & W_{4j}\left[S_-(t)\cos(\psi_j^0) + S_+(t)\sin(\psi_j^0)\right] T_-(t) \\ & - W_{3j}\left[S_-(t)\cos(\psi_j^0) - S_+(t)\sin(\psi_j^0)\right] T_+(t)\big\} \\ & + \epsilon\,(-1)^j W_{12}\sqrt{(\bar{I}_1 + \hat{I}_1)(\bar{I}_2 + \hat{I}_2)} \lim_{n\to\infty} \int_{\frac{-2\pi n}{\omega_{12}}}^{\frac{2\pi n}{\omega_{12}}} \mathrm{d}t\, \sin\left[\omega_{12} t + (\psi_1^0 - \psi_2^0)\right]. \end{aligned} \tag{28}$$

After integrating once by parts and some rearrangement we are left with

$$\begin{aligned} M_j^+(\psi_j^0) = \lim_{n\to\infty} \epsilon\,\frac{1}{\omega_j}\sqrt{\bar{I}_j + \hat{I}_j} \int_{\frac{-2\pi n}{\omega_j}}^{\frac{2\pi n}{\omega_j}} \mathrm{d}t\, \sin(\omega_j t) \\ \times \big\{\left[W_{4j}\,(S_-' T_- + S_- T_-') - W_{3j}\,(S_-' T_+ + S_- T_+')\right]\cos(\psi_j^0) \\ + \left[W_{4j}\,(S_+' T_- + S_+ T_-') + W_{3j}\,(S_+' T_+ + S_+ T_+')\right]\sin(\psi_j^0)\big\} \end{aligned} \tag{29}$$

where the primed quantities denote the time derivative of S and T given explicitly by

$$S'_{\pm}[p_h(t)] = -\frac{p_h(t)}{2}\frac{1}{S_{\pm}}\left[\frac{1}{\sqrt{N^2-(p_h(t))^2}} \pm \frac{\gamma}{2V}\right]\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} \tag{30}$$

$$T'_{\pm}[p_h(t)] = \pm\tfrac{1}{2}N\frac{T_{\pm}}{N^2-(p_h(t))^2}\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} \tag{31}$$

$$\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = -\left(\frac{8V^2}{\gamma}\right)(\alpha-1)\frac{\sinh(2V\sqrt{\alpha-1}t)}{\cosh^2(2V\sqrt{\alpha-1}t)}. \tag{32}$$

Since $S'_{\pm}$ and $T'_{\pm}$ are rapidly decreasing expressions, we are allowed to extend the integration limits in (28) to infinity and thus obtain convergent integrals. In summary, we find that the components of the Melnikov vector are of the form

$$M_j^{+}(\psi_j^0) = F_j(\bar{I};\gamma,V,W_{3j},W_{4j})\cos(\psi_j^0) + G_j(\bar{I};\gamma,V,W_{3j},W_{4j})\sin(\psi_j^0) \tag{33}$$

where F_j and G_j, $(j = 1,2)$ are functions of the system parameters determined via the integrals in (29).

The components of the Melnikov vector have simple zeros at

$$\psi_j^0 = \arctan[-(F_j/G_j)] + k\pi \qquad k = 0,\pm1,\pm2,\ldots \tag{34}$$

and because of

$$\det\left[D_{(\psi_1^0,\psi_2^0)}M_{\bar{I}}(\psi_1^0,\psi_2^0)\right] = \sum_{j=1,2}\left[G_j\cos(\psi_j^0) - F_j\sin(\psi_j^0)\right] \neq 0 \tag{35}$$

$[D_{(\psi_1^0,\psi_2^0)}M_{\bar{I}}]$ has rank 2 at the zero set of $M_{\bar{I}}$. Due to the theorems (4.1.19) and (4.1.20) in [24] the stable and unstable manifolds, $W^{s,u}(\mathcal{T}_\epsilon)(\bar{I})$, of the KAM torus intersect transversely near (ψ_1^0,ψ_2^0) in the five-dimensional energy surface.

In summary, on each five-dimensional energy manifold we have a one-parameter family of 2-tori. The tori along with their three-dimensional stable and unstable manifolds are not isolated on the energy manifold. In the perturbed system most of these tori survive and their stable and unstable manifolds intersect transversely. The resulting homoclinic tangles of the manifolds of a surviving KAM torus can intersect the tangles of manifolds of neighbouring KAM tori, establishing a transition chain. In this way orbits may wander among the KAM tori in a chaotic fashion called Arnold diffusion [31]. As a consequence, motion of arbitrarily high period can be found which, for instance, might be reflected in an irregular energy transfer among various nonlinear vibrational modes of a molecular tetramer system.

5. Summary

In this paper an analytical proof is given for the non-integrability of the DNLS-tetramer dynamics. We have applied Hamiltonian methods such as Marsden–Weinstein reduction in combination with Wiggins' generalized Melnikov technique. Starting from the DNLS dimer as the underlying integrable subsystem the coupling to the additional tetramer sites has been treated perturbationally. The system in the complex amplitudes has been reduced to a real-valued Hamiltonian system with three degrees of freedom. In analysing the geometrical structure of the reduced phase space special attention was paid to the homoclinic manifold. We established the presence of chaotic dynamics in the form of Arnold diffusion by means of the generalized Melnikov method. The existence of an infinite number of zeros of the components of the Melnikov function was shown, which in turn guarantees the presence of homoclinic tangles around the homoclinic manifold implying the existence of Arnold

diffusion. The DNLS tetramer can also be regarded as a finite segment embedded in a longer extended DNLS chain. Since chaotic motion is present in the $n = 4$ subsystem it will very likely also occur in an extended chain.

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